The Special Declination Problem

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May 10, 1997

The problem below was raised by Abouanass Mohamed, an email correspondent. It is pleasing that the solution expression turns out to be so simple and compact.

Suppose that the hour angle of a star is h_a at altitude a, and h_b at altitude b. The problem is to determine the star's declination δ which minimizes (or maximizes) $h_a - h_b$. This is a generalization of the problem of finding when the shortest (or the longest) twilight occurs. For example, if a is set to -18 (the sun's altitude at the end of astronomical twilight) and b to 0, then the appropriate value of the sun's declination determines the date of the shortest (or the longest) duration of astronomical twilight.

We start with the following notation:

$\delta =$ Star's declination

$h_x =$ Star's hour angle when its altitude is x

$\phi =$ Observer's latitude

The hour angle is given by the equation:

$$\cos h_x = \frac{\sin x - \sin\phi\sin\delta}{\cos\phi\cos\delta}$$

Therefore

$$\frac{d(h_x)}{d\delta} = \frac{-1}{\sqrt{1 - (\frac{\sin x - \sin\phi\sin\delta}{\cos\phi\cos\delta})^2}} \frac{d}{d\delta}(\frac{\sin x}{\cos\phi}\frac{1}{\cos\delta} - \frac{\sin\phi}{\cos\phi}\tan\delta)$$

After some simplification, we get

$$\frac{d(h_x)}{d\delta} = \frac{\sin\phi - \sin x\sin\delta}{\cos\delta\sqrt{\cos^2\phi - \sin^2\delta - \sin^2 x + 2\sin x\sin\phi\sin\delta}}$$

To compute the value of δ which minimizes (or maximizes) $h_a - h_b$, the condition is

$$0 = \frac{d(h_a - h_b)}{d\delta} = \frac{d(h_a)}{d\delta} - \frac{d(h_b)}{d\delta} = \frac{P(a)}{\cos\delta\sqrt{Q(a)}} - \frac{P(b)}{\cos\delta\sqrt{Q(b)}}$$

where

$$P(x) = \sin\phi - \sin x\sin\delta$$
$$Q(x) = \cos^2\phi - \sin^2\delta - \sin^2 x + 2\sin x\sin\phi\sin\delta$$

That is,

$$\frac{P(a)}{\cos\delta\sqrt{Q(a)}} = \frac{P(b)}{\cos\delta\sqrt{Q(b)}}$$

or

$$P(a)^2Q(b) - P(b)^2Q(a) = 0$$

This reduces to

$$-\sin^4\delta(\sin^2 a - \sin^2 b)$$

$$
\begin{gathered}
+2\sin^3\delta\sin\phi(\sin a\cos^2 b-\cos^2 a\sin b)\\
+\sin^2\delta\cos^2\phi(\sin^2 a-\sin^2 b)\\
-2\sin\delta\sin\phi(\sin a\cos^2 b-\cos^2 a\sin b)\\
+(\sin^2 a-\sin^2 b)\sin^2\phi\\
=0
\end{gathered}
$$

After replacing all $\cos^2 X$ by $1-\sin^2 X$, and doing some simplification, the LHS is seen to have a factor $\sin a-\sin b$. Since $a\neq b$, we remove this factor to get

$$
\begin{gathered}
-\sin^4\delta(\sin a+\sin b)\\
+2\sin^3\delta\sin\phi(1+\sin a\sin b)\\
+\sin^2\delta(1-\sin^2\phi)(\sin a+\sin b)\\
-2\sin\delta\sin\phi(1+\sin a\sin b)\\
+\sin^2\phi(\sin a+\sin b)\\
=0
\end{gathered}
$$

Collecting degree 4, 2, 0 terms (in $\sin\delta$) and degree 3, 1 terms separately, we have

$$
\begin{gathered}
-(\sin^4\delta-\sin^2\delta+\sin^2\delta\sin^2\phi-\sin^2\phi)(\sin a+\sin b)\\
+2\sin\delta(\sin^2\delta-1)\sin\phi(1+\sin a\sin b)\\
=0
\end{gathered}
$$

There is a common factor $\sin^2\delta-1$ which corresponds to $\delta=\pm\pi/2$. Removing the factor, we get a quadratic equation

$$
\sin^2\delta(\sin a+\sin b)-2\sin\delta\sin\phi(1+\sin a\sin b)+\sin^2\phi(\sin a+\sin b)=0
$$

whose roots are

$$\sin\delta = \frac{\sin\phi(1+\sin a\sin b) \pm \sqrt{\sin^2\phi(1+\sin a\sin b)^2 - \sin^2\phi(\sin a+\sin b)^2}}{\sin a+\sin b}$$

That is,

$$\sin\delta = \sin\phi\frac{1+\sin a\sin b \pm \cos a\cos b}{\sin a+\sin b}$$

By using angle-sum and half-angle trigonometric formulas, the solution simplifies to

$$\sin\delta = \sin\phi\frac{\sin\frac{a+b}{2}}{\cos\frac{a-b}{2}}$$

and

$$\sin\delta = \sin\phi\frac{\cos\frac{a-b}{2}}{\sin\frac{a+b}{2}}$$