Computing the Qibla Direction

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1 Introduction

The qibla direction at any point on the earth's surface, assuming the earth to be a perfect sphere, is given by the great circle passing through that point and the Kaaba at Makkah. Of the two arcs of this great circle, the qibla direction is along the shorter arc from the given point to the Kaaba. The angle between the true north at a given location and the qibla direction at that location can be obtained by solving the spherical triangle whose vertices are the North Pole, the Kaaba, and the given location. For a detailed description of the qibla concept, including some historical background and references, see [1].

This article describes how to compute the qibla direction for any location on Earth. It also describes how to determine the qibla using shadows, a method which does not require knowing the compass directions or measuring any angles. The qibla-related problems discussed in this article are essentially problems of solving spherical triangles. So the article starts by summarizing the needed solution methods.

2 Solution of Spherical Triangles

A *great circle* on a sphere is a circle that has the same center and same radius as the sphere itself. Equivalently, a great circle on a sphere is the line of intersection of the sphere and a plane passing through the center of the sphere. An arc of a great circle is measured by the angle that the arc subtends at the center of the sphere. The angle between two great circle arcs is measured by the angle between the planes of the great circles containing those two arcs.

A *spherical triangle* is a figure on a sphere enclosed by three great circle arcs. These arcs and their points of intersection are called the *sides* and *vertices* of the spherical triangle. The spherical triangles that we use will have angles and sides measuring at most 180° in absolute value.

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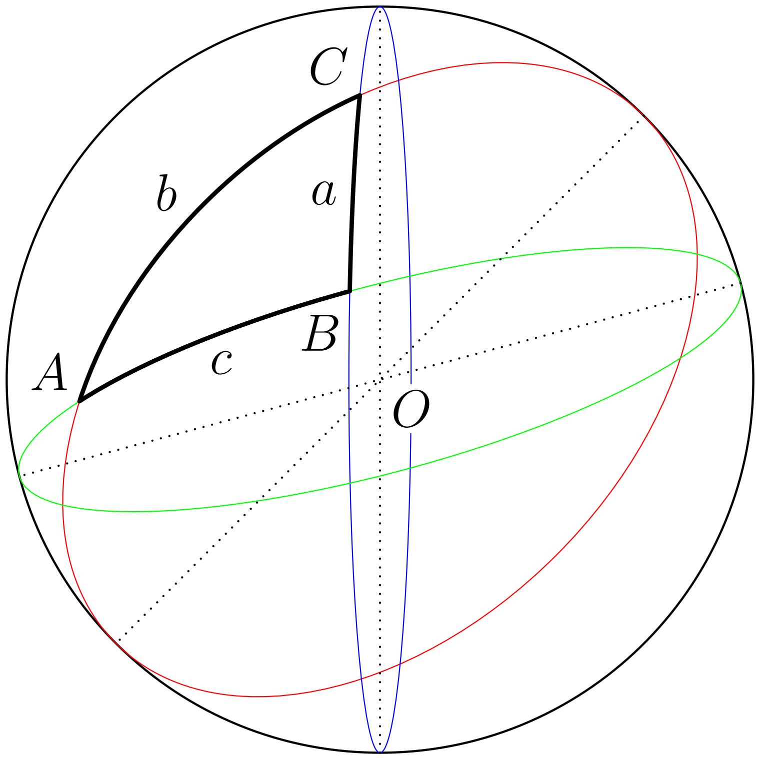


Figure 1: A spherical triangle

Figure 1 shows a sphere, its center O, and three great circles drawn on it in color. These great circles intersect pairwise on points A, B, and C, so the figure ABC is a spherical triangle with vertices A, B, and C, and sides AB, BC, and CA. The measures of angles and sides are typically indicated, respectively, by the symbols of vertices in uppercase and the symbols of the opposite vertices in lowercase. So the angles and sides of this triangle are A, B, C, a, b, and c, as marked on the figure.

Solving a spherical triangle refers to finding the values of some specified sides and/or angles of the triangle when the values of certain other sides and/or angles are given. Each set of given and required quantities is referred to as a *case*. The example of a case is: "Given two sides and the angle opposite to one of those sides, find the angle included by those sides."

2.1 Solutions of Certain Cases

This section describes the cases relevant to our computations. Not all the unknowns in these cases are needed for our problems, but they are required sometimes to help resolve ambiguities in solutions.

The equations used in our solutions are all derived from what are often termed "fundamental formulas of spherical trigonometry" describing the relationships among the angles and sides of spherical triangles. These formulas are stated and proved in spherical trigonometry textbooks. An excellent account of spherical trigonometry is included in W.M. Smart's book

[2] on astronomy, available for free legal download from the Web. A particularly pleasing summary of spherical trigonometry with several helpful hints is in J. Tatum's online book [3] on celestial mechanics.

The formulas that we make use of are listed below:

The sine formula:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{1}$$

The cosine formulas:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{2}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{3}$$

The cotangent formula:

$$\cos b \cos C = \sin b \cot a - \sin C \cot A. \tag{4}$$

Napier's analogies:

$$\tan \frac{a+b}{2} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \frac{c}{2}, \tag{5}$$

$$\tan \frac{A+B}{2} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{C}{2}. \tag{6}$$

Other formulas similar to the above (or, "companion" formulas) are obtained by permuting the side and angle symbols as follows: $a \to b \to c \to a$ and $A \to B \to C \to A$.

2.1.1 Case I: Given three sides, find one of the angles.

Given sides a, b, and c, the formula for angle A is:

$$A = \cos^{-1} \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{7}$$

This formula follows directly from Equation (2). The formulas for angles B and C can be obtained by suitably changing the letters in the above formula.

2.1.2 Case II: Given two sides and their included angle, find (i) the angle opposite to one of the given sides and (ii) the third side.

Given a, b, and C, the formulas for A and c are:

$$A = \tan^{-1} \frac{\sin C}{\sin b \cot a - \cos b \cos C}, \tag{8}$$

$$c = \cos^{-1}(\cos a \cos b + \sin a \sin b \cos C). \tag{9}$$

These formulas (8) and (9) are derived from Equations (4) and (3), respectively. The inverse tangent function $\tan^{-1}$ results in an angle value in the $-90°$ to $90°$ range. For our spherical triangle solutions, we need angle values in the 0° to $180°$ range. The function `atan2` that is available in most programming languages produces precisely such results. The equivalent of $\tan^{-1}(x/y)$ is `atan2(x,y)`. It also takes care of the situation in which the denominator y evaluates to zero. For example, while $\tan^{-1}(1/0)$ and $\tan^{-1}(-1/0)$ would fail during evaluation, `atan2(1,0)` and `atan2(-1,0)` would result in the correct angles equal to $90°$ and $-90°$, respectively, when converted to degrees. So in a computer program, it's best to use the following equivalent of Equation (8):

$$A = \operatorname{atan2}(\sin C, \sin b \cot a - \cos b \cos C). \tag{10}$$

2.1.3 Case III: Given two sides and the angle opposite to one of those sides, find (i) the third side and (ii) the angle included by the given sides.

Given a, b, and A, the solution for c and C proceeds best by first finding the other opposite angle B:

$$B = \sin^{-1} \frac{\sin A \sin b}{\sin a}, \tag{11}$$

$$c = 2\tan^{-1} \frac{\cos \frac{A+B}{2} \tan \frac{a+b}{2}}{\cos \frac{A-B}{2}}, \tag{12}$$

$$C = 2\tan^{-1} \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2} \tan \frac{A+B}{2}}. \tag{13}$$

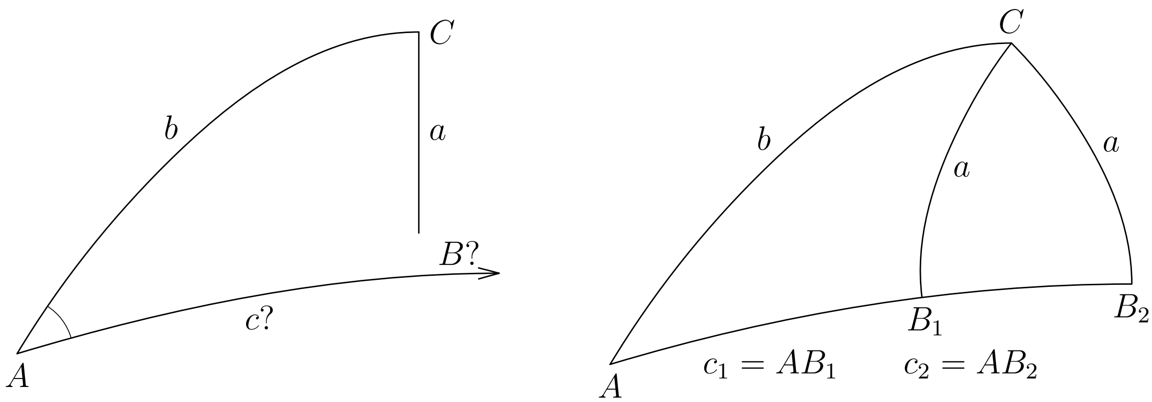
These three formulas are derived from Equations (1), (5), and (6), in that order. Again, as discussed in Case II, Equations (12) and (13) are best processed in computer programs by the following equivalents:

$$c = 2\operatorname{atan2}(\cos((A+B)/2)\tan((a+b)/2), \cos((A-B)/2)), \tag{14}$$

$$C = 2\operatorname{atan2}(\cos((a-b)/2), \cos((a+b)/2)\tan((A+B)/2)). \tag{15}$$

If in Equation (11), the expression $(\sin A \sin b)/(\sin a)$ has magnitude larger than 1, then it cannot be the sine of any angle, and B is undefined. On the other hand, since $\sin(x) = \sin(180° - x)$, if Equation (11) is satisfied by some value B_1, then it is also satisfied by another value $180° - B_1$ (unless $B_1 = 90°$, and both values are the same). Equation (11) can thus produce zero, one, or two different values of B. If Equation (11) has two solutions for B, they each have to be tried in determining C and c. So, in general, Case III can yield zero, one, or two spherical triangles as solutions.

Note: For a value of B to be a valid solution, it must also satisfy the condition that the quantities $a - b$ and $A - B$ have the same sign (positive, negative, or zero.)



(a) No Solution

(b) Two Solutions

Figure 2: Case III, non-unique solutions

Figure 2 illustrates the subcases of Cases III with zero and two solutions. Sub-figure (a) is an example of the given side a being too small to go with the given side b and angle A in order for a spherical triangle to materialize. On the other hand, Sub-figure (b) is an example of two different spherical triangles satisfying the given values of sides and angle a, b, and A.

3 Qibla Computation

In computing the qibla direction, we use the following *sign convention*:

- Latitudes are positive if north and negative if south.
- Longitudes are positive if east and negative if west.
- The qibla angle is measured eastward or westward from north. Its magnitude is between 0 and 180 degrees, and its sign is positive for directions east of north and negative for directions west of north.

3.1 Basic Qibla Formula

We refer to the spherical triangle of Figure 3 as the *Qibla Triangle*. In this triangle, K represents the Kaaba, N represents the North Pole, and A represents the given location whose qibla direction is to be computed. The great circle arcs AN and KN are along the meridians (great-semicircles of constant longitudes) through A and K, respectively. At the end A, the arc AN points to the north. The qibla is along the great circle arc AK. The spherical angle $q = NAK$ is the angle at A from the north direction AN to the direction AK towards the Kaaba, and so q is the angle to be computed. Let ϕ and λ be the latitude and longitude of A, and ϕ_K and λ_K be the latitude and longitude of K (the Kaaba). If all angles and arc-lengths are measured in degrees, then the arcs AN and KN are of lengths $90° - \phi$ and $90° - \phi_K$, respectively. Also, the angle ANK between the meridians of K and A equals the difference between the longitudes of K and A, that is, $\lambda_K - \lambda$. Here we are given two sides and the included angle of a spherical triangle, and it is required to determine one other angle.

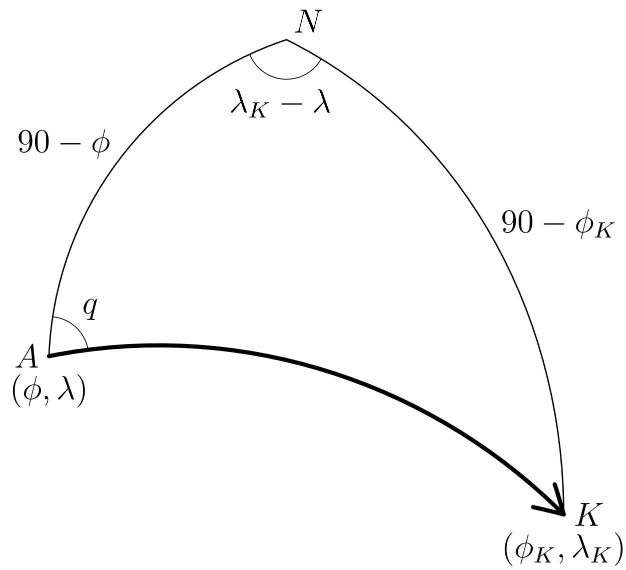


Figure 3: The Qibla Triangle

This is an example of Case II described in Section 2.1.2, and Equation (8) applies directly. The symbols of Equation (8) correspond to elements of Figure 3 as follows: $a \to 90° - \phi_K$, $b \to 90° - \phi$, $C \to \lambda_K - \lambda$, and $A \to q$. So the solution for q is:

$$q = \tan^{-1} \frac{\sin(\lambda_K - \lambda)}{\cos\phi \tan\phi_K - \sin\phi \cos(\lambda_K - \lambda)}. \tag{16}$$

3.2 Computational Details

Some care has to be exercised in using Equation (16). According to our convention for specifying the qibla direction, we would like the angle q (measured from the true North direction) to be in the range $-180°$ to $180°$, that is, in any of the four quadrants. But the $\tan^{-1}$ function produces the result in the range $-90°$ to $90°$, that is, restricted to two quadrants only. So if the computation is being undertaken on a simple hand-calculator, then the correct quadrant of q has to be selected manually such that $\sin(q)$ and $\cos(q)$ have the same sign as the numerator and denominator of Equation (16).

As already mentioned in Sections 2.1.2 and 2.1.3, most programming languages provide a two-argument mathematical function `atan2` that can be used to implement Equation (16) as follows:

$$q = \text{atan2}(sin(\lambda_K - \lambda), cos(\phi) * tan(\phi_K) - sin(\phi) * cos(\lambda_K - \lambda)).$$

With the signs of latitudes and longitudes chosen according to the previously stated convention, the qibla angle computed by this code automatically has the absolute value between 0 and 180 degrees, and the sign exactly according to the above convention. So the computed qibla angle is in the correct quadrant.

When the angle $\lambda_K - \lambda$ (the difference between the longitudes of the given location and the Kaaba) turns out to have magnitude greater than 180°, it indicates that the arc AK is the larger arc of the great circle through the given location and the Kaaba. In this case, we need to switch to the shorter arc by replacing the angle by $360° - (\lambda_K - \lambda)$ and reversing the direction. In other words, we need to compute

$$q = -\tan^{-1} \frac{\sin(360° - (\lambda_K - \lambda))}{\cos\phi \tan\phi_K - \sin\phi \cos(360° - (\lambda_K - \lambda))}. \tag{17}$$

But Equation (16) produces exactly the same answer. So, when using Equation (16), the check whether the longitude difference exceeds 180° in magnitude is, fortunately, unnecessary.

3.3 Qibla Determination Using A Calculator

Most scientific calculators lack the `atan2` function. So if we wish to compute the qibla using a calculator, then we have to go through a rather tedious process to derive the answer in the correct quadrant. Specifically, first we have to check whether the answer is one of certain special values. If that is not the case, then we have to adjust the value computed from Equation (16) by examining the signs of the numerator and the denominator of the fraction on the right-hand side of the equation. The computational procedure is as follows:

1. If the numerator is zero, then q is zero. (Ignore the value of the denominator.)
2. If the denominator is zero, then q depends on the numerator, as follows:
   (a) If the numerator is positive, then q is 90°.
   (b) If the numerator is negative, then q is −90°.
3. Compute q using equation (16).
4. If the denominator is positive, then leave q unchanged.
5. If the denominator is negative, then depending on the sign of the numerator, make the following changes:
   (a) If the numerator is positive, add 180° to q.
   (b) If the numerator is negative, subtract 180° from q.

After these adjustments, the magnitude of q will be between −180° and 180°, and its sign will be positive for directions east of north and negative for directions west of north.

3.4 Examples

The examples below illustrate qibla computation using Equation (16) on a calculator which provides $\tan^{-1}$ but not `atan2` or $\cot^{-1}$. Together, these examples cover qibla directions in all four quadrants.

We use the following values for the geographical coordinates of the Kaaba: latitude $\phi_K = 21°25'21''\text{N} = +21°.422500$ and longitude $\lambda_K = 39°49'34''\text{E} = +39°.826111$.

To find the qibla for (1) Washington, D.C.; (2) Anchorage, Alaska; (3) Ankara, Turkey; and (4) Tashkent, Uzbekistan.

(1) For *Washington, D.C.*, we use the coordinates: $\phi = 38°54'\text{N} = +38°.9$, $\lambda = 77°01'\text{W} = -77°.016667$. Substituting these values in Equation (16), we get the value of the numerator as $+0.892249$, the denominator as $+0.588896$, and q as $+56.574700$. Since the denominator is positive, no adjustment is needed, and we obtain $q = +56°.574700 = 56°34'$ E of N. This is a northeastern direction.

(2) For *Anchorage, Alaska*, we use the coordinates: $\phi = 61°13'\text{N} = +61°.216667$, $\lambda = 149°53'\text{W} = -149°.883333$. Substituting these values in Equation (16), we get the value of the numerator as -0.168652, the denominator as $+1.052810$, and q as -9.101030. Again, since the denominator is positive, no adjustment is needed, and we obtain $q = -9°.101030 = 9°06'$ W of N. This is a northwestern direction.

(3) For *Ankara, Turkey*, we use the coordinates: $\phi = 39°55'\text{N} = +39°.916667$, $\lambda = 32°50'\text{E} = +32°.833333$. Substituting these values in Equation (16), we get the value of the numerator as $+0.121744$, the denominator as -0.335977, and q as -19.918500. Since the denominator is negative, an adjustment to q is needed. As the numerator is positive, we need to add 180° to q, getting $q = +160°.081500 = 160°05'$ E of N. This is a southeastern direction.

(4) For *Tashkent, Uzbekistan*, we use the coordinates: $\phi = 41°16'\text{N} = 41°.266667$, $\lambda = 69°15'\text{E} = +69°.25000$. Substituting these values in Equation (16), we get the value of the numerator as -0.491267, the denominator as -0.279578, and q as $+60.355900$. Since the denominator is negative, an adjustment to q is needed. As the numerator is also negative, we need to subtract 180° from q, getting $q = -119°.644000 = -119°39'$ W of N. This is a southwestern direction.

4 Qibla Computation Using Shadows

To determine the qibla, not only we need to compute the value of the qibla angle accurately but also need to know the north direction from which this angle has to be measured. It is important to note that the qibla angle computed as above is to be measured from the *true* (or *geographic*) north that is, the direction of the earth's geographic north pole.

When a magnetic compass is used to determine the qibla, two angular measurements have to be made. First, since the true or geographic north is different from the *magnetic* north, an angular correction, called the *magnetic declination* has to be applied. (Note: The terms "magnetic declination" and "declination" should not be confused with each other. The latter term, used in many places in this paper, is an astronomical term describing how high above the celestial equator a heavenly object is at any time.) Then the qibla angle has to be measured from the north so found. Both these operations are prone to angular measurement errors. The main problem is extending the very short length of the magnetic needle. GPS-based electronic compasses give the true north directly, eliminating the need to correct for magnetic deviation. Yet these compasses do not eliminate the angular measurement errors since the arrows on these compasses are also short lines.

A method that bypasses these problems altogether is to observe the shadow of a vertical object at a time when the shadow makes an easily measurable angle such as 0°, 90°, 180°, or 270° with the direction of the qibla. The angles 0° and 180° imply that the qibla is exactly in the same direction as, or exactly opposite to, the shadow, so at these times there is no angle to measure and the qibla can be found most accurately. The other two angles imply that the qibla is perpendicular to the shadow, so again the qibla can be found quite accurately. Note that one does not need the knowledge of the north direction (whether true or magnetic) when using this method. Depending upon the season, there may or may not be such a time during a given day when the shadow is in the qibla direction. But usually times with other bearings, such as 90° or 270°, can be found almost every day.

We will describe how to compute the times on any given date in any given location when the shadow of a vertical object forms a specified angle with the qibla. But this turns out to be an application of the problem of computing the time when the sun appears in a specified direction on any given day in a given location. So we deal with that more general problem first.

4.1 The Astronomical Triangle

For computing the time when the sun appears in a specified direction on any given day in a given location, it's best to use a spherical triangle in which the vertices are the projections of the location, the earth's north pole, and the sun on the *celestial sphere*. Such a spherical triangle (involving any heavenly body, not just the sun) is called an *astronomical triangle*. Figure 4 shows the astronomical triangle PZX for our problem. It's drawn on the celestial sphere whose center O coincides with the center of the earth. The points P and Q represent the *celestial north and south poles*, which are the projections of the corresponding terrestrial poles, and are obtained by extending the earth's diameter passing through the terrestrial poles to the celestial sphere. The point Z is the *zenith* which is the point in the celestial sphere vertically above the (observer in a) given location. The point X is the position of a heavenly body (the sun—in our present case) in the celestial sphere.

The circle $NWSE$ represents the projection of the *horizon* on the celestial sphere. The line OZ (not shown in Figure 4) is perpendicular to the plane of $NWSE$, and the points N, W, S,

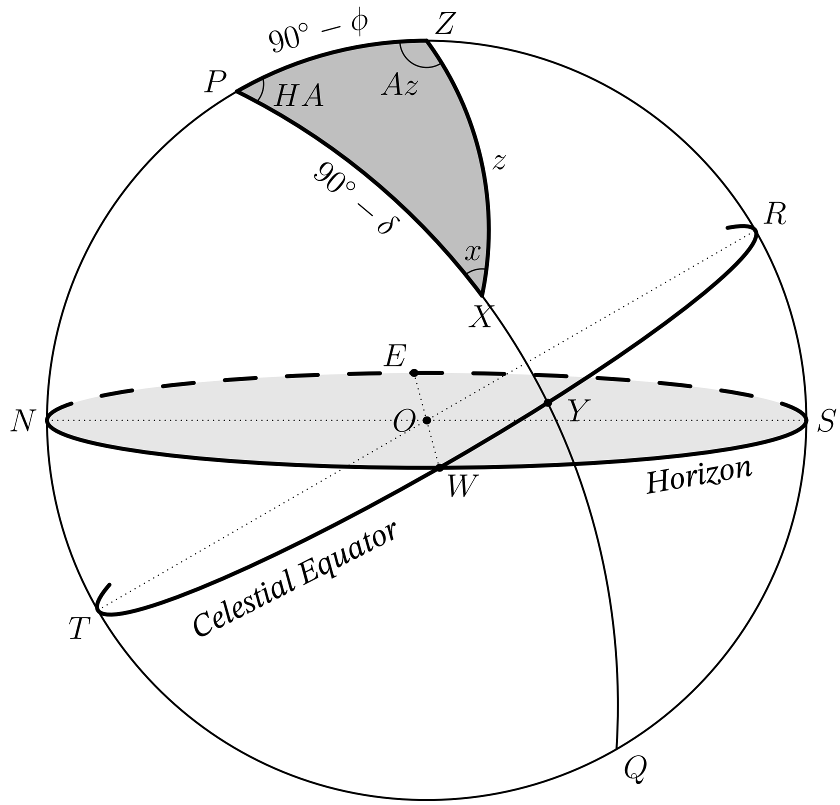


Figure 4: The astronomical triangle

and E represent, respectively, the north, west, south, and east points on the celestial sphere—often referred to as the *cardinal points*. The circle TWR represents the *celestial equator* which is the projection of the earth's equator onto the celestial equator. Note that the equator and horizon intersect in the points E and W. The outer circle in Figure 4 represents the great circle on the celestial sphere passing through the zenith Z, the poles P and Q, and the cardinal points N and S. This is often called the observer's *vertical meridian*.

The sides and angles of the astronomical triangle PZX have the following significance:

1. As the points P and Z on the vertical meridian $PZRQ$ are projections of the terrestrial north pole and the given location, respectively, and the point R is on the celestial equator, the great circle arc RZ equals the observer's latitude, say, ϕ, and hence the arc PZ equals $90° - \phi$. The arc PZ is often referred to as the location's *co-latitude*.

2. PXQ is the semi-great circle passing through the sun X and the celestial poles P and Q. Let Y be the point of intersection of PXQ and the celestial equator TWR. The *declination* δ of a heavenly body is its distance from the celestial equator. The sun's declination is therefore the great circle arc XY, and hence the great circle arc PX equals $90° - \delta$. The arc PX is often referred to as the sun's *co-declination*.

3. The side ZX of the astronomical triangle is called the *zenith distance*, and we use z to denote the arc length of ZX. If the arc ZX is extended to cut the horizon $NWSE$ in a point

U (not shown in Figure 4), then the angular measure of the arc UX would represent how high above the horizon the sun would appear in the sky. This angle is called the sun's *altitude*, and it thus equals $90° - z$. The arc ZX is, therefore, often referred to as the sun's *co-altitude*.

4. Since Z and P are the celestial circle projections, respectively, of the given location and the terrestrial north pole, the angle PZX equals the horizontal angle that the direction to the heavenly body would make with the north direction. This angle is called the *azimuth*, and we use Az to denote its angular measure. So, for a heavenly body, the altitude is the vertical angle above the horizon, while the azimuth is the horizontal angle from the north direction.

5. The angle XPZ is called the *hour angle*, and we use HA to denote its measure. The reason for calling it the hour angle is as follows: The time when the sun crosses the vertical meridian (which contains the pole P and the observer's zenith Z) is the *solar noon* (or *local time of noon*) at the observer's location. The angle XPZ therefore represents the angle through which the sun has to move since noon to attain its position at X. If this angle (XPZ) is converted from degrees to time ($360° = 24$ hours—based on the sun's apparent movement around the earth), it is a measure of the time interval elapsed since noon for the sun to be at X.

6. The angle PXZ is not needed by itself, but it can appear as an intermediate value in some calculations. We use x to denote its angular measure.

4.2 Solving the Astronomical Triangle

Before addressing the shadow/qibla application, we solve the general problem of computing, for any given location on any given date, the time when the sun appears in a desired direction. For this we need to solve the spherical triangle ZPX with the known and unknown elements as follows:

Known elements:

1. The location is given, so its latitude ϕ is known. Hence the side PZ is known.

2. The azimuth Az, that is, the angle PZX, is also known since it is precisely the direction in which the sun is to appear.

3. The side PX depends on the value of the sun's declination δ. The declination for any longitude on the earth at any time on a given date can be computed from astronomical equations. A comprehensive source of astronomical algorithms is the book [4] by J. Meeus. The value of sun's declination for any date and time can also be obtained from an online solar position calculator such as [5]. Of course, we don't know the time yet, because it is what we trying to determine, so we cannot compute the declination at that exact moment. However, the sun's declination changes so slowly that for the precision

that we require (the time to be accurate to about a minute), we can consider the declination δ at the desired (unknown) time on the given day to be practically the same as the declination at some fixed time on that day. For example, we can use the declination value at noon. Or, to be a bit more accurate, at 9 AM, say, for a morning phenomenon and at 3 PM for an afternoon phenomenon. The solar declination at noon can also simply be looked up in an astronomical table. So, to a sufficient approximation, the side PX is also known. (Once the time has been approximated, we can iterate through the steps again by computing the declination at that time, then recomputing the time with this new value of declination. But this is totally unnecessary for the precision we want.)

Unknown element:

For our spherical triangle solution problem, the only unknown component is the hour angle HA, that is, XPZ. However, once HA is found, there still remains the task of determining from it the time of the day. The shadow time computation example will show the steps for that task.

4.3 Solution Method

This is the case of solving a spherical triangle when two of its sides and the angle opposite one of them is given, and the angle included by the two given sides is required. It's discussed in Section 2.1.3, and Equations (11) and (13) (equivalently, (15)) apply here.)

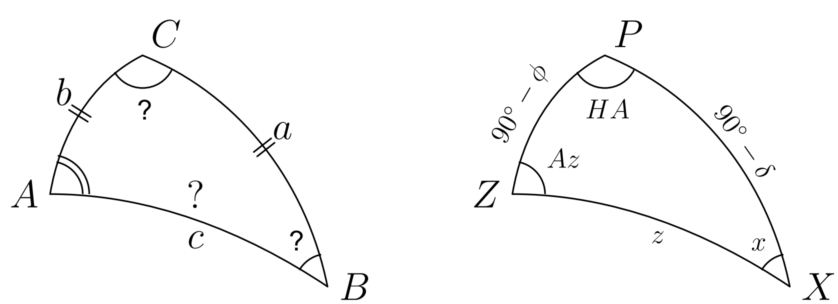


Figure 5: Solving the astronomical triangle

The general spherical triangle of Figure 1 and the astronomical triangle of Figure 4 are drawn together in Figure 5. As displayed in this figure, the correspondence between the elements of the two triangles is:

Vertices: $A \to Z$, $B \to X$, and $C \to P$.

Sides: $a \to 90° - \delta$, $b \to 90° - \phi$, and $c \to z$.

Angles: $A \to Az$, $B \to x$, and $C \to HA$.

These symbol correspondences are precisely the substitutions for side and angle symbols that we can make in Equations (11) and (13) (or (15)) of the Section 2.1.3. So the steps for our needed solution are:

1. Compute δ for a fixed time on the given date, say, noon, 9 AM, or 3 PM, using the equations give in [4]. Or, obtain its value from an online solar calculator such as [5].

2. The above-listed substitutions turn Equation (11) into the following equation for computing the angle x:
$$x = \sin^{-1} \frac{\sin Az \cos \phi}{\cos \delta}. \tag{18}$$
As discussed in Section 2.1.3, it is possible for this equation to have zero, one, or two solutions for x. First, the fraction on the right-hand side of this equation can have a value larger than 1. In that case, the equation is not satisfiable, and we conclude that at no time on the given date would the sun attain the desired direction. It is also possible for two different values of x to satisfy this equation. If the two values are x_1 and x_2 (which equals $180° - x_1$), then the steps below should be carried out for each of them.

3. The above substitutions turn Equation (12) into the following equation for computing the side z:
$$z = 2\tan^{-1} \frac{\cos\frac{1}{2}(Az + x)\cot\frac{1}{2}(\phi + \delta)}{\cos\frac{1}{2}(Az - x)}. \tag{19}$$
Or, to make use of the `atan2` function with Equation (14),
$$z = 2\,\text{atan2}(\cos((Az + x)/2)\cot((\phi + \delta)/2), \cos((Az - x)/2)). \tag{20}$$
The value of z, the zenith distance, is needed if the sun is to be in the given direction *during daylight hours*. (The sun might get to the given direction during the nighttime and be below the horizon!) At the time of sunset, the sun's zenith distance is maximum and equals 90°.83, so we must have $z \leq 90°.83$ for the sun to be *actually visible* in the given direction! In that case, we proceed to the next step to compute HA.

4. The above substitutions turn Equation (13) into the following equation for computing the angle HA:
$$HA = 2\tan^{-1} \frac{\cos\frac{1}{2}(\phi - \delta)}{\sin\frac{1}{2}(\phi + \delta)\tan\frac{1}{2}(Az + x)}. \tag{21}$$
Or, to make use of the `atan2` function with Equation (15),
$$HA = 2\,\text{atan2}(\cos((\phi - \delta)/2), \cos((\phi + \delta)/2)\tan((Az + x)/2)). \tag{22}$$

The spherical triangle solution is now complete since the hour angle HA has been determined. However, once HA is found, there still remains the task of determining from it the time of the day when the sun appears in the direction specified by the given azimuth. For that we need two more data items:

1. The **Equation of Time** for the given date is another item which either needs to be computed using equations from astronomy (e.g., in [4]) or obtained from an online solar position calculator, e.g., [5].

2. The **longitude of standard time** for the given location. Its value in degrees is *standard time in hours* ×15. So, for example, since Karachi's standard time is +5h, its longitude of standard time is 75°.

The shadow time computation example in the next section will show how the values of sun's hour angle, the local time of noon, and the longitude of standard time are used to determine the time when the sun attains the given azimuth.

4.4 Computing the Time for a Desired Shadow Direction

We assume that angles are positive when measured clockwise and negative when measured anti-clockwise. (This is consistent with the convention that the qibla angle measured from the north, and is positive when eastward and negative when westward.)

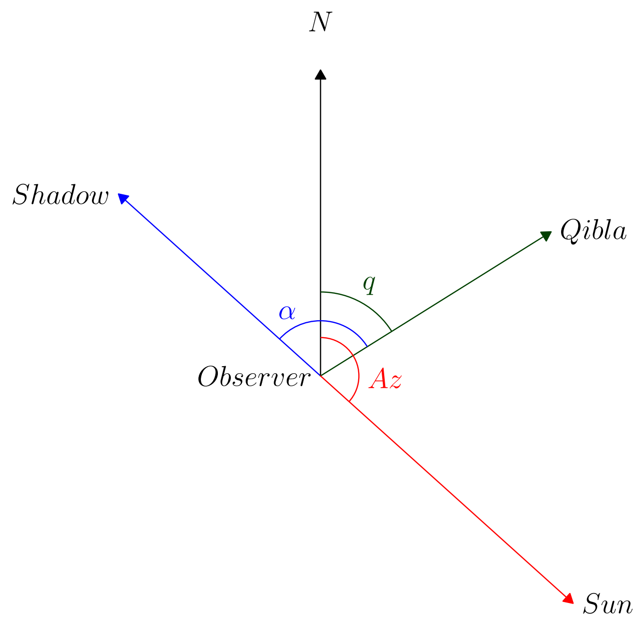


Figure 6: Shadow and Qibla

Refer to Figure 6. Let q be the qibla angle for the given location, and let α, $0° \leq \alpha < 360°$, be the desired angle measured clockwise from the shadow to the qibla. Then the angle that the shadow bears to the north is $q - \alpha$. Adding 180° to the shadow's angle from the north gives the sun's angle from the north. So the sun's azimuth is $q - \alpha + 180°$ at the time that we are looking for. Here is a procedure to compute the time of the day when the shadow forms a specified angle with the qibla. To make the procedure self-contained, the symbols are being re-listed.

Notation

ϕ = latitude of place, λ = longitude of place, λ_s = longitude of standard time,
q = clockwise angle from true North to the qibla,

α = clockwise angle from shadow to the qibla,
Az = Sun's azimuth, x = auxiliary angle, z = Sun's zenith distance, HA = Sun's hour angle,
δ = Sun's declination, EoT =Equation of Time in hours,
t = (the desired answer:) standard time of correct shadow direction.

Data

$\phi, \lambda, \lambda_s, \alpha, q$ given.
δ, EoT for the given date computed or obtained separately.

Procedure

1. Compute Az by $Az = |q - \alpha + 180° \pm 360°k|$ with k any integer so that $0° \leq Az < 360°$. If now $Az \geq 180°$ then reassign $Az = 360° - Az$ and note that this shadow, if it occurs, will occur as a PM (afternoon) phenomenon. Otherwise, keep Az unchanged and note that this shadow is an AM (morning) phenomenon.

2. Compute x from (18) reproduced below

$$x = \sin^{-1} \frac{\sin Az \cos \phi}{\cos \delta}.$$

   This equation might yield zero, one, or two solutions. If the fraction on the right-hand side has a magnitude larger than zero, then there are no solutions. But as the Note in Section 2.1.3 mentions, there is a further check to see if either or both computed values of x are acceptable solutions. In a spherical triangle, the larger of any two sides is opposite to the vertex with the larger angle. The angles x and Az are opposite, respectively, to the sides $90° - \phi$ and $90° - \delta$. So, for an x to be acceptable we must have

$$x \gtreqless Az \quad \text{if} \quad (90 - \phi) \gtreqless (90 - \delta). \text{ Or,}$$
$$x \gtreqless Az \quad \text{if} \quad \delta \gtreqless \phi.$$

   This condition must be checked for each value of x satisfying the above equation before going on to the next steps.

3. Compute z from (19) or (20. In case the previous step produced two solutions for x, let the corresponding values of z be z_1 and z_2. If both z_1 and z_2 are greater than 90°.83, then stop; the phenomenon does not occur during daylight hours. That is, at no time on the given day does the shadow form the specified angle with the qibla.

4. If both z_1, z_2 are less than or equal to 90°.83, then both are suitable values of z. But the larger one provides the time of more accurate observation because the shadow is longer at that time. Choose that one as z. If only one value is less or equal to 90°.83, choose that one as z.

5. Compute HA from (21) or (22).

6. Compute t from

$$t = 12 - EoT + \frac{\lambda - \lambda_s + HA}{15} \text{ for PM time (shadow towards the east).} \quad (23)$$

$$t = 12 - EoT + \frac{\lambda - \lambda_s - HA}{15} \text{ for AM time (shadow towards the east).} \quad (24)$$

Convert t to hours and minutes.

Examples

For Washington, DC, find the times, if any, on November 1, 2021 at which the shadow of a vertical object will be
(a) in the direction of the qibla, and
(b) at 90° angle anti-clockwise from the qibla.

The qibla for Washington, DC, was already calculated in Example (1) of Section 3.4. From there we take these values:
latitude, $\phi = 38°54'N = +38°.9$; longitude, $\lambda = 77°01'W = -77°.0167$; and
the qibla angle, $q = 56°.5747$ (East of North).
Furthermore, we assume that the following two values have been independently determined for the given date:
Equation of Time $EoT = 16.47$ minutes $= 0.2745$ hours; and solar declination at noon time $\delta = -14°.66$.

(a) For the specified shadow bearing, $\alpha = 0°$.

$Az = |56°.5747 - 0° + 180° \pm 0 \times 360°| = 236°.57$. Change the value of Az to $360° - 236°.57$, i.e., $123°.43$, and note that this is a PM phenomenon.

From Equation (18),

$$x = \sin^{-1} \frac{\sin 123°.43 \cos 38°.9}{\cos(-14°.66)}$$
$$= 42°.1710 \text{ or } 137°.8290.$$

For checking the values of x for being acceptable, we compare δ and ϕ. As $\delta = -14°.66$ is less than $\phi = 38°.9$, x must be less than $Az = 123°.43$. Hence we reject $x = 137°.829$ and keep only $x = 42°.171$ as the acceptable solution.

From Equation (19),

$$z = 2\tan^{-1} \frac{\cos\frac{1}{2}(123°.43 + 42.171)\cot\frac{1}{2}(38°.9 - 14°.66)}{\cos\frac{1}{2}(123°.43 - 42.171)}$$
$$= 75°.1178$$

From Equation (21),

$$HA = 2\tan^{-1}\frac{\cos\frac{1}{2}(38^\circ.9 + 14^\circ.66)}{\sin\frac{1}{2}(38^\circ.9 - 14^\circ.66)\tan\frac{1}{2}(123^\circ.43 + 42.171)}$$
$$= 56^\circ.4815$$

As we have a PM phenomenon here, we apply Equation (23).

$$t = 12 - 0.2745 + \frac{77.0167 - 75 + 56.4815}{15} = 15.6254$$
$$= 3{:}38 \text{ PM.}$$

(b) For the specified shadow direction, $\alpha = -90^\circ$.

$Az = |56^\circ.5747 - 90^\circ + 180^\circ + 0 \times 360^\circ| = 146^\circ.57.$ This is an AM phenomenon

From Equation (18),

$$x = \sin^{-1}\frac{\sin 146^\circ.57 \cos 38^\circ.9}{\cos(-14^\circ.66)}$$
$$= 26^\circ.3032 \text{ or } 153^\circ.6968.$$

For checking the values of x for being acceptable, we compare δ and ϕ. As $\delta = -14^\circ.66$ is less than $\phi = 38^\circ.9$, x must be less than $Az = 123^\circ.43$. Hence we reject $x = 153^\circ.6968$ and keep only $x = 26^\circ.3032$ as the acceptable solution.

From Equation (19),

$$z = 2\tan^{-1}\frac{\cos\frac{1}{2}(146^\circ.57 + 26^\circ.3032)\cot\frac{1}{2}(38^\circ.9 - 14^\circ.66)}{\cos\frac{1}{2}(146^\circ.57 - 26^\circ.3032)}$$
$$= 60^\circ.330$$

From Equation (21),

$$HA = 2\tan^{-1}\frac{\cos\frac{1}{2}(38^\circ.9 + 14^\circ.66)}{\sin\frac{1}{2}(38^\circ.9 - 14^\circ.66)\tan\frac{1}{2}(146^\circ.57 + 26^\circ.3032)}$$
$$= 29^\circ.6614$$

As we have an AM phenomenon, we apply Equation (24).

$$t = 12 - 0.2745 + \frac{77.0167 - 75 - 29.6614}{15} = 9.883$$
$$= 9{:}53 \text{ AM.}$$

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